TWMS J. Pure Appl. Math., V.XX, N.XX, 20XX, pp.XX-XX

CLASSIFICATION OF A FAMILY OF THREE DIMENSIONAL REAL EVOLUTION ALGEBRAS

A.N. IMOMKULOV[1]

Abstract. In this paper we classify a family of three-dimensional real evolution algebras. We also consider an evolution operator for an evolution algebra and find fixed points of this operator for two and three-dimensional cases. Then we construct an evolution algebra, the matrix of structural constants of which is Jacobian of the evolution operator at a fixed point. We study isomorphism between these evolution algebras.

Keywords: Evolution algebra, evolution operator, isomorphism, Jacobian, fixed point.

AMS Subject Classification: 13J30, 13M05.

1. Introduction

To study a non-linear function one usually finds the linear approximation to the function at a given point. The linear approximation of a function is the first order Taylor expansion around the point of interest. In the theory of dynamical systems, linearization is a method for assessing the local stability of an equilibrium point. For an algebra (with a fixed multiplication $*$) and the (cubic) matrix $\mathcal{M}$ of structural constants one can define a quadratic (non-linear) operator $F(x) = x*x$, with coefficients given by the matrix $\mathcal{M}$. The Jacobian J of F at a given point, can be considered as a linear approximation of F. Consequently, J generates an evolution algebra as its matrix of structural constants.

Let $(A, \cdot)$ be an algebra over a field K. If it admits a countable basis $e_1, e_2, \dots, e_n, \dots$, such that

$$e_i \cdot e_j = 0, \ if \ i \neq j$$

$$e_i \cdot e_i = \sum_k a_{ik} e_k, \ for \ any \ i$$

then it is called an evolution algebra. This basis is called a natural basis.

We note that to every evolution algebra corresponds a square matrix (a_{ik}) of structural constants of the given evolution algebra.

In [10] the following basic properties of evolution algebras are proved:

1) Evolution algebras are not associative, in general.
2) Evolution algebras are commutative, flexible.
3) Evolution algebras are not power-associative, in general.
4) The direct sum of evolution algebras is also an evolution algebra.

In [7] the dynamics of absolutely nilpotent and idempotent elements in chains generated by two-dimensional evolution algebras are studied. In [2] the authors consider an evolution algebra which has a rectangular matrix of structural constants. This algebra is called evolution algebras of "chicken" population (EACP). The mentioned paper is devoted to the description

[1]Institute of Mathematics, Tashkent, Uzbekistan,
e-mail: aimomkulov@gmail.com

Manuscript received XXXXXXXX 20XX.

of structure of EACPs. Using the Jordan form of the rectangular matrix of structural constants, a simple description of EACPs over the field of complex numbers is given. The classification of three-dimensional complex EACPs is obtained. Moreover, some $(n+1)$-dimensional EACPs are described. The fundamentals of evolution algebras have been being developed in the last years with no probabilistic restrictions on the stucture constants [4, 5, 8].

In Section 2 we study an approximation of two-dimensional real evolution algebras and isomorphism between these algebras. In Section 3 we will classify a family of three-dimensional real evolution algebras. We show that there are 13 class of such evolution algebras. We consider an approximation of three-dimensional real evolution algebras in Section 4 and also study isomorphism between these algebras.

2. Approximation of two-dimensional real evolution algebras

Let E be a 2-dimensional evolution algebra over the field of real numbers. Such algebras are classified in [6]:

Theorem 2.1. [6] *Any two-dimensional real evolution algebra E is isomorphic to one of the following pairwise non-isomorphic algebras:*

(i) $dim(E^2)=1$:

E_1: $e_1e_1=e_1$, $e_2e_2=0$;

E_2: $e_1e_1=e_1$, $e_2e_2=e_1$;

E_3: $e_1e_1=e_1+e_2$, $e_2e_2=-e_1-e_2$;

E_4: $e_1e_1=e_2$, $e_2e_2=0$;

E_5: $e_1e_1=e_2$, $e_2e_2=-e_2$;

(ii) $dim(E^2)=2$:

$E_6(a_2;a_3)$: $e_1e_1=e_1+a_2e_2$, $e_2e_2=a_3e_1+e_2$; $1-a_2a_3\neq 0$, $a_2,a_3\in\mathbb{R}$. *Moreover* $E_6(a_2;a_3)$ *is isomorphic to* $E_6(a_3;a_2)$.

$E_7(a_4)$: $e_1e_1=e_2$, $e_2e_2=e_1+a_4e_4$, *where* $a_4\in\mathbb{R}$;

For a given evolution algebra $(E,\cdot)$ an evolution operator has the following form $F(x)=x\cdot x=x^2$. If $x=\sum_{i=1}^n x_ie_i$ then

$$x^2=\sum_{i=1}^n x_i^2e_i^2=\sum_{i=1}^n x_i^2\Big(\sum_{k=1}^n a_{ik}e_k\Big)=\sum_{k=1}^n\Big(\sum_{i=1}^n a_{ik}x_i^2\Big)e_k.$$

We denote $x'_k=\sum_{i=1}^n a_{ik}x_i^2$. Thus we have the following operator, $F:E\to E$,

$$F: x'_k=\sum_{i=1}^n a_{ik}x_i^2, k=\overline{1,n}.$$

Jacobian of the operator F at the point x for two-dimensional case has a form

$$J_F(x)=\begin{pmatrix} 2a_{11}x_1 & 2a_{21}x_2\\ 2a_{12}x_1 & 2a_{22}x_2\end{pmatrix}.$$

Following [9] and [3] we define an evolution algebra $\widetilde{E}$ with matrix $J_F(x)$ as the matrix of structural constants.

We will find fixed points of this operator, i.e. solutions of $F(x)=x$:

$$\begin{cases} x_1=a_{11}x_1^2+a_{21}x_2^2,\\ x_2=a_{12}x_1^2+a_{22}x_2^2.\end{cases} \tag{1}$$

Note that $(0,0)$ is one of solutions of system of equations (1), and

$$J_F(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

So corresponding evolution algebra with matrix $J_F(0,0)$ is trivial.

Trivial evolution algebras are not interesting. So we will find a non-zero solutions denoted by $(x_1^0; x_2^0)$ of (1) for algebras $E_i,\ i = 1, 2, \ldots, 7$ mentioned in Theorem 2.1 and study isomorphisms of evolution algebras corresponding to these fixed points with other evolution algebras.

In the following table we give all possibilities for two-dimensional case:

2-dimensional real evolution algebras	Non-zero real fixed points of the operator F	Corresponding evolution algebras to fixed points
$E_1 : \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	$(1;0)$	$\widetilde{E}_1 : \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$
$E_2 : \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$	$(1;0)$	$\widetilde{E}_2 : \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$
$E_3 : \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$	Not exists	Not exists
$E_4 : \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$	Not exists	Not exists
$E_5 : \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$	$(0;-1)$	$\widetilde{E}_5 : \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$
$E_6(a_2;a_3) : \begin{pmatrix} 1 & a_2 \\ a_3 & 1 \end{pmatrix}$	$(x_1^0;x_2^0)$	$\widetilde{E}_6(a_2;a_3) : \begin{pmatrix} 2x_1^0 & 2a_3x_2^0 \\ 2a_2x_1^0 & 2x_2^0 \end{pmatrix}$
$E_7(a_4) : \begin{pmatrix} 0 & 1 \\ 1 & a_4 \end{pmatrix}$	$(x_1^0;x_2^0)$ if $a_4 \geq -\frac{3}{\sqrt[3]{4}}$	$\widetilde{E}_7(a_4) : \begin{pmatrix} 0 & 2x_2^0 \\ 2x_1^0 & 2a_4x_2^0 \end{pmatrix}$

We have the following theorem.

Theorem 2.2. *i) Evolution algebras $\widetilde{E}_1$, $\widetilde{E}_2$ and $\widetilde{E}_5$ are isomorphic to E_1;*
ii) $\widetilde{E}_6(a_2;a_3)$ is isomorphic to the evolution algebra $E_6(b_2;b_3)$, where $b_2 = a_3\left(\frac{x_2^0}{x_1^0}\right)^2$, $b_3 = a_2\left(\frac{x_1^0}{x_2^0}\right)^2$;
iii) $\widetilde{E}_7(a_4)$ is isomorphic to $E_7(b_4)$, where $b_4 = a_4\sqrt[3]{(\frac{x_2^0}{x_1^0})^2}$.

Proof. $\widetilde{E}_1 \approxeq E_1$: By the change of basis $\widetilde{e}_1 = \frac{1}{2}e_1$ we can prove that the evolution algebra $\widetilde{E}_1$ is isomorphic to E_1.
$\widetilde{E}_2 \approxeq E_1$: It is similar to the above proof.
$\widetilde{E}_5 \approxeq E_1$: By the change of basis $\widetilde{e}_1 = \frac{1}{2}e_2$, $\widetilde{e}_2 = e_1$ we can prove that the evolution algebra $\widetilde{E}_5$ is isomorphic to E_1.
$\widetilde{E}_6(a_2;a_3) \approxeq E_6(b_2;b_3)$: We can see this by the change of basis $\widetilde{e}_1 = \frac{1}{2x_1^0}e_1$ and $\widetilde{e}_2 = \frac{1}{2x_2^0}e_2$.
$\widetilde{E}_7(a_4) \approxeq E_7(b_4)$: We can see this by the change of basis $\widetilde{e}_1 = 2\sqrt[3]{x_1^0(x_2^0)^2}e_1$ and $\widetilde{e}_2 = 2\sqrt[3]{(x_1^0)^2x_2^0}e_2$. □

3. Three-dimensional real evolution algebras with $dim(E^2) = 1$

In [1] three dimensional complex evolution algebras are classified. Now we shall consider classification of three dimensional real evolution algebras.

Fix a three-dimensional real evolution algebra E and a natural basis $B = \{e_1, e_2, e_3\}$. Let M_B be the matrix of structural constants of E relative to B:

$$M_B = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

In order to classify three dimensional real evolution algebras with condition $dim(E^2) = 1$ we find a basis of E for which its structure matrix has an expression as simple as possible, where by 'simple' we mean with the maximal number of 0, 1 and -1 in the entries.

Let $dim(E^2) = 1$. Without loss of generality we may assume $e_1^2 \neq 0$. Write $e_1^2 = a_1e_1 + a_2e_2 + a_3e_3$, where $a_i \in \mathbb{R}$ and $a_i \neq 0$ for some i. Note that e_1^2 is basis of E^2.

Since $e_2^2, e_3^2 \in E^2$, there exist $c_1, c_2 \in \mathbb{R}$ such that

$$e_2^2 = c_1e_1^2 = c_1(a_1e_1 + a_2e_2 + a_3e_3),$$
$$e_3^2 = c_2e_1^2 = c_2(a_1e_1 + a_2e_2 + a_3e_3).$$

Then

$$M_B = \begin{pmatrix} a_1 & a_2 & a_3 \\ c_1a_1 & c_1a_2 & c_1a_3 \\ c_2a_1 & c_2a_2 & c_2a_3 \end{pmatrix}.$$

We analyze when E^2 has the extension property. This means that there exists a natural basis $B' = \{e_1', e_2', e_3'\}$ of E with

$$\begin{aligned} e_1' &= e_1^2 = a_1e_1 + a_2e_2 + a_3e_3 \\ e_2' &= \alpha e_1 + \beta e_2 + \gamma e_3 \\ e_3' &= \delta e_1 + \nu e_2 + \eta e_3 \end{aligned} \tag{2}$$

for some $\alpha, \beta, \gamma, \delta, \nu, \eta \in \mathbb{R}$ such that $\beta\eta - \gamma\nu \neq 0$. This implies that

$$|P_{B'B}| = \begin{vmatrix} a_1 & a_2 & a_3 \\ \alpha & \beta & \gamma \\ \delta & \nu & \eta \end{vmatrix} \neq 0. \tag{3}$$

By products $e_1'e_2' = 0, e_1'e_3' = 0, e_2'e_3' = 0$, B' is a natural basis if and only if the following conditions are satisfied:

$$\alpha a_1 + \beta a_2c_1 + \gamma a_3c_2 = 0 \tag{4}$$
$$\delta a_1 + \nu a_2c_1 + \eta a_3c_2 = 0 \tag{5}$$
$$\alpha\delta + \beta\nu c_1 + \gamma\eta c_2 = 0.$$

In the above conditions, the structure matrix of E relative to B' is:

$$M_{B'} = \begin{pmatrix} a_1^2 + a_2^2c_1 + a_3^2c_2 & 0 & 0 \\ \alpha^2 + \beta^2c_1 + \gamma^2c_2 & 0 & 0 \\ \delta^2 + \nu^2c_1 + \eta^2c_2 & 0 & 0 \end{pmatrix}.$$

Now, we start with the analysis of possible cases.

Case 1. Suppose that $a_1 \neq 0$.
By changing the basis, we may assume that $e_1^2 = e_1 + a_2e_2 + a_3e_3$. Using (4) we get $\alpha = -(\beta a_2c_1 + \gamma a_3c_2)$ and by (5), $\delta = -(\nu a_2c_1 + \eta a_3c_2)$. If we replace α and δ in (3) we obtain that:

$$|P_{B'B}| = (1 + a_2^2c_1 + a_3^2c_2)(\beta\eta - \gamma\nu).$$

Now we check that $|P_{B'B}|$ is zero or not. This happens depending on $1 + a_2^2c_1 + a_3^2c_2$ being zero or not.

Case 1.1 Assume $1 + a_2^2 c_1 + a_3^2 c_2 = 0$.
In this case E^2 has not the extension property since $|P_{B'B}| = 0$. We will analyze what happens when $1 + a_3^2 c_2 \neq 0$ and when $1 + a_3^2 c_2 = 0$.

Case 1.1.1 If $1 + a_3^2 c_2 \neq 0$.
Note that $a_2^2 c_1 \neq 0$ since otherwise we get a contradiction. Then $c_1 = \frac{-1-a_3^2 c_2}{a_2^2}$. In this case, the structure matrix is:

$$M_B = \begin{pmatrix} 1 & a_2 & a_3 \\ \frac{-1-a_3^2 c_2}{a_2^2} & \frac{-1-a_3^2 c_2}{a_2} & \frac{(-1-a_3^2 c_2)a_3}{a_2^2} \\ c_2 & c_2 a_2 & c_2 a_3 \end{pmatrix}.$$

Case 1.1.1.1 Suppose that $a_3 \neq 0$.
If we take the natural basis $B'' = \{e_1, a_2 e_2, a_3 e_3\}$, then

$$M_{B''} = \begin{pmatrix} 1 & 1 & 1 \\ -1 - a_3^2 c_2 & -1 - a_3^2 c_2 & -1 - a_3^2 c_2 \\ a_3^2 c_2 & a_3^2 c_2 & a_3^2 c_2 \end{pmatrix}. \tag{6}$$

We are going to verify two cases: $c_2 = 0$ and $c_2 \neq 0$.

Assume first $c_2 = 0$. Then $M_{B''} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$. By considering another change of basis we find a structure matrix with more zeros. Namely, let $B''' = \{e_2, e_1 + e_3, e_3\}$. Then

$$M_{B'''} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In what follows we will assume that $c_2 \neq 0$. We recall that we are considering the structure matrix given in (6). Take $I := < (1 + a_3^2 c_2)e_1 + e_2 >$. Then I is a two-dimensional evolution ideal which is degenerate as an evolution algebra.

Now, for B''' the natural basis change is

$$P_{B'''B''} = \begin{pmatrix} \frac{1+D}{2(1+a_3^2 c_2)} & \frac{-1+D}{2(1+a_3^2 c_2)} & \frac{1+D}{2(1+a_3^2 c_2)} \\ \frac{-1+D}{2(1+a_3^2 c_2)} & \frac{1+D}{2(1+a_3^2 c_2)} & \frac{-1+D}{2(1+a_3^2 c_2)} \\ -(a_3^2 c_2) & 0 & 1 \end{pmatrix}$$

where $D = (a_3^2 c_2)^3 + 2(a_3^2 c_2)^2 + (a_3^2 c_2)$ and we obtain:

$$M_{B'''} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix}.$$

Note that $|P_{B'''B''}| = -2(a_3^2 c_2)(1 + a_3^2 c_2)^2 \neq 0$ because $a_3^2 c_2 \neq 0$ and $a_3^2 c_2 \neq -1$.

Case 1.1.1.2 Suppose that $a_3 = 0$. Then $1 + a_2^2 c_1 = 0$ and necessarily $a_2^2 c_1 \neq 0$. In this case,

$$M_B = \begin{pmatrix} 1 & a_2 & 0 \\ \frac{-1}{a_2^2} & \frac{-1}{a_2} & 0 \\ c_2 & c_2 a_2 & 0 \end{pmatrix}. \tag{7}$$

Again we will verify two cases depending on c_2.

Assume $c_2 > 0$. Take $B'' = \{e_1, a_2e_2, \frac{1}{\sqrt{c_2}}e_3\}$. Then $M_{B''} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$, which has already appeared.

Assume $c_2 < 0$. Take $B'' = \{e_1, a_2e_2, \frac{1}{\sqrt{-c_2}}e_3\}$. Then $M_{B''} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ -1 & -1 & 0 \end{pmatrix}$.

Suppose $c_2 = 0$. Then, for $B'' = \{e_1, a_2e_2, e_3\}$ we have $M_{B''} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, matrix that has already appeared.

Case 1.1.2 Suppose that $1 + a_3^2c_2 = 0$.
This implies that $a_3^2c_2 \neq 0$ and $a_2^2c_1 = 0$.

Case 1.1.2.1 Assume $c_1 \neq 0$.
This implies that $a_2 = 0$. Moreover, since $a_3 \neq 0$, necessarily $c_2 = \frac{-1}{a_3^2}$. If we take the natural basis $B'' = \{e_1, e_3, e_2\}$, then $M_{B''} = \begin{pmatrix} 1 & a_3 & 0 \\ \frac{-1}{a_3^2} & \frac{-1}{a_3} & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and we are in the Case 1.1.1.2.

Case 1.1.2.2 Suppose $c_1 = 0$ and $a_2 = 0$.
Take $B'' = \{e_1, a_3e_3, e_2\}$. Then $M_{B''} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ as above.

Case 1.1.2.3 Suppose $c_1 = 0$ and $a_2 \neq 0$.
Taking $B'' = \{e_1, e_3, e_2\}$, we are in the same conditions as in the Case 1.1.1.1 with $c_2 = 0$.

Case 1.2 Assume $1 + a_2^2c_1 + a_3^2c_2 \neq 0$.
We will prove that E^2 has the extension property. In any subcase we will provide with a natural basis for E one of its elements gives a natural basis of E^2.

Case 1.2.1 Suppose that $c_1 = c_2 = 0$.
Consider the natural basis $B' = \{e_1^2, e_2 + e_3, 2e_2 + e_3\}$. Then

$$M_{B'} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We claim that this evolution algebra does not have a two-dimensional evolution ideal generated by one element. To prove this consider $f = me_1 + ne_2 + pe_3$. Then the ideal I generated by f is the linear span of $\{f\} \cup \{m^ie_i\}_{i\in\mathbb{N}}$. In order for I to have a natural basis with two elements, necessarily $m = 0$, implying that the dimension of I is one, a contradiction.

Case 1.2.2 Assume that $c_1 = 0$ and $c_2 \neq 0$.
Then $1 + c_2a_3^2 \neq 0$. For $B' = \{e_1 + a_2e_2 + a_3e_3, e_2, -a_3c_2e_1 + e_2 + e_3\}$ the structure matrix is

$$M_{B'} = \begin{pmatrix} 1 + c_2a_3^2 & 0 & 0 \\ 0 & 0 & 0 \\ c_2(1 + c_2a_3^2) & 0 & 0 \end{pmatrix}.$$

Note that E^2 has the extension property because the first element in B' is e_1^2, which is a natural basis of E^2.

Case 1.2.2.1 Assume that $c_2 > 0$.
Consider $B'' = \left\{\frac{1}{1+c_2a_3^2}e_1, e_2, \frac{1}{\sqrt{c_2}(1+c_2a_3^2)}e_3\right\}$. Then

$$M_{B''} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Case 1.2.2.2 Assume that $c_2 < 0$.
Consider $B'' = \left\{\frac{1}{1+c_2a_3^2}e_1, e_2, \frac{1}{\sqrt{-c_2}(1+c_2a_3^2)}e_3\right\}$. Then

$$M_{B''} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

We claim that these evolution algebras do not have a two-dimensional evolution ideal generated by one element. Let $f = \alpha e_1 + \beta e_2 + \gamma e_3$. Then the ideal generated by f, say I, is the linear span of $\{f, \gamma e_1, \alpha e_1\} \cup \{(\alpha^2+\gamma^2)\alpha^i e_1\}_{i\in\mathbb{N}\cup\{0\}} \cup \{(\alpha^2+\gamma^2)^2\alpha^i e_1\}_{i\in\mathbb{N}\cup\{0\}}$. After some computations, in order for I to have dimension 2 and to be degenerated implies $\alpha = 0$ or $\gamma = 0$, a contradiction.

Case 1.2.3 If $c_1 > 0$ and $c_2 > 0$.

If B' is the natural basis such that $P_{B'B} = \begin{pmatrix} 1 & a_2 & a_3 \\ -a_2c_1 & 1 & 0 \\ \frac{-a_3c_2}{1+c_1a_2^2} & \frac{-a_3a_2c_2}{1+c_1a_2^2} & 1 \end{pmatrix}$, we obtain that $M_{B'} =$

$$\begin{pmatrix} 1+a_2^2c_1+a_3^2c_2 & 0 & 0 \\ c_1(1+c_1a_2^2) & 0 & 0 \\ \frac{c_2(1+a_2^2c_1+a_3^2c_2)}{(1+c_1a_2^2)} & 0 & 0 \end{pmatrix}.$$

Now, consider the natural basis $B'' = \{f_1, f_2, f_3\}$ such that

$$P_{B''B'} = \begin{pmatrix} \frac{1}{1+a_2^2c_1+a_3^2c_2} & 0 & 0 \\ 0 & \frac{1}{\sqrt{c_1(1+c_1a_2^2)(1+a_2^2c_1+a_3^2c_2)}} & 0 \\ 0 & 0 & \frac{\sqrt{1+c_1a_2^2}}{\sqrt{c_2}(1+a_2^2c_1+a_3^2c_2)} \end{pmatrix}$$

and the structure matrix is

$$M_{B''} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

It is not difficult to show that this evolution algebra does not have a degenerate two-dimensional evolution ideal generated by one element.

Case 1.2.4 If $c_1 > 0$ and $c_2 < 0$.

For the natural basis B' such that $P_{B'B} = \begin{pmatrix} 1 & a_2 & a_3 \\ -a_2c_1 & 1 & 0 \\ \frac{-a_3c_2}{1+c_1a_2^2} & \frac{-a_3a_2c_2}{1+c_1a_2^2} & 1 \end{pmatrix}$, we obtain that $M_{B'} =$

$$\begin{pmatrix} 1+a_2^2c_1+a_3^2c_2 & 0 & 0 \\ c_1(1+c_1a_2^2) & 0 & 0 \\ \frac{c_2(1+a_2^2c_1+a_3^2c_2)}{(1+c_1a_2^2)} & 0 & 0 \end{pmatrix}.$$

Case 1.2.4.1 Assume $1+a_2^2c_1+a_3^2c_2>0$.
Now, consider the natural basis $B''=\{f_1,f_2,f_3\}$ such that

$$P_{B''B'}=\begin{pmatrix} \frac{1}{1+a_2^2c_1+a_3^2c_2} & 0 & 0 \\ 0 & \frac{1}{\sqrt{c_1(1+c_1a_2^2)(1+a_2^2c_1+a_3^2c_2)}} & 0 \\ 0 & 0 & \frac{\sqrt{1+c_1a_2^2}}{\sqrt{-c_2}(1+a_2^2c_1+a_3^2c_2)} \end{pmatrix}$$

and the structure matrix is

$$M_{B''}=\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Case 1.2.4.2 Assume $1+a_2^2c_1+a_3^2c_2<0$.
Now, consider the natural basis $B''=\{f_1,f_2,f_3\}$ such that

$$P_{B''B'}=\begin{pmatrix} \frac{1}{1+a_2^2c_1+a_3^2c_2} & 0 & 0 \\ 0 & \frac{1}{\sqrt{-c_1(1+c_1a_2^2)(1+a_2^2c_1+a_3^2c_2)}} & 0 \\ 0 & 0 & \frac{\sqrt{1+c_1a_2^2}}{\sqrt{-c_2}(1+a_2^2c_1+a_3^2c_2)} \end{pmatrix}$$

and the structure matrix is:

$$M_{B''}=\begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Case 1.2.5 If $c_1<0$ and $c_2>0$.

Case 1.2.5.1 Assume $1+a_2^2c_1>0$.

For the natural basis B' such that $P_{B'B}=\begin{pmatrix} 1 & a_2 & a_3 \\ -a_2c_1 & 1 & 0 \\ \frac{-a_3c_2}{1+c_1a_2^2} & \frac{-a_3a_2c_2}{1+c_1a_2^2} & 1 \end{pmatrix}$, we obtain that $M_{B'}=$

$$\begin{pmatrix} 1+a_2^2c_1+a_3^2c_2 & 0 & 0 \\ c_1(1+c_1a_2^2) & 0 & 0 \\ \frac{c_2(1+a_2^2c_1+a_3^2c_2)}{(1+c_1a_2^2)} & 0 & 0 \end{pmatrix}.$$

Case 1.2.5.1.1 Assume $1+a_2^2c_1+a_3^2c_2>0$.
Now, consider the natural basis $B''=\{f_1,f_2,f_3\}$ such that

$$P_{B''B'}=\begin{pmatrix} \frac{1}{1+a_2^2c_1+a_3^2c_2} & 0 & 0 \\ 0 & \frac{1}{\sqrt{-c_1(1+c_1a_2^2)(1+a_2^2c_1+a_3^2c_2)}} & 0 \\ 0 & 0 & \frac{\sqrt{1+c_1a_2^2}}{\sqrt{c_2}(1+a_2^2c_1+a_3^2c_2)} \end{pmatrix}$$

and the structure matrix is:

$$M_{B''} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

It is not difficult to show that this evolution algebra does not have a degenerate two-dimensional evolution ideal generated by one element.

Case 1.2.5.1.2 Assume $1 + a_2^2 c_1 + a_3^2 c_2 < 0$.
Now, consider the natural basis $B'' = \{f_1, f_2, f_3\}$ such that

$$P_{B''B'} = \begin{pmatrix} \frac{1}{1+a_2^2c_1+a_3^2c_2} & 0 & 0 \\ 0 & \frac{1}{\sqrt{c_1(1+c_1a_2^2)(1+a_2^2c_1+a_3^2c_2)}} & 0 \\ 0 & 0 & \frac{\sqrt{1+c_1a_2^2}}{\sqrt{c_2}(1+a_2^2c_1+a_3^2c_2)} \end{pmatrix}$$

and the structure matrix is: $M_{B''} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, which has already appeared.

Case 1.2.5.2 Assume $1 + a_2^2 c_1 < 0$.

For B' the natural basis such that $P_{B'B} = \begin{pmatrix} 1 & a_2 & a_3 \\ -a_2c_1 & 1 & 0 \\ \frac{-a_3c_2}{1+c_1a_2^2} & \frac{-a_3a_2c_2}{1+c_1a_2^2} & 1 \end{pmatrix}$, we obtain that $M_{B'} =$

$$\begin{pmatrix} 1 + a_2^2c_1 + a_3^2c_2 & 0 & 0 \\ c_1(1 + c_1a_2^2) & 0 & 0 \\ \frac{c_2(1+a_2^2c_1+a_3^2c_2)}{(1+c_1a_2^2)} & 0 & 0 \end{pmatrix}.$$

Case 1.2.5.2.1 Assume $1 + a_2^2 c_1 + a_3^2 c_2 > 0$.
Now, consider the natural basis $B'' = \{f_1, f_2, f_3\}$ such that

$$P_{B''B'} = \begin{pmatrix} \frac{1}{1+a_2^2c_1+a_3^2c_2} & 0 & 0 \\ 0 & \frac{1}{\sqrt{c_1(1+c_1a_2^2)(1+a_2^2c_1+a_3^2c_2)}} & 0 \\ 0 & 0 & \frac{\sqrt{-(1+c_1a_2^2)}}{\sqrt{c_2}(1+a_2^2c_1+a_3^2c_2)} \end{pmatrix}$$

and the structure matrix is: $M_{B''} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$, which has already appeared.

Case 1.2.5.2.2 Assume $1 + a_2^2 c_1 + a_3^2 c_2 < 0$.
Now, consider the natural basis $B'' = \{f_1, f_2, f_3\}$ such that

$$P_{B''B'} = \begin{pmatrix} \frac{1}{1+a_2^2c_1+a_3^2c_2} & 0 & 0 \\ 0 & \frac{1}{\sqrt{-c_1(1+c_1a_2^2)(1+a_2^2c_1+a_3^2c_2)}} & 0 \\ 0 & 0 & \frac{\sqrt{-(1+c_1a_2^2)}}{\sqrt{c_2}(1+a_2^2c_1+a_3^2c_2)} \end{pmatrix}$$

and the structure matrix is: $M_{B''} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$, which has already appeared.

Case 1.2.6 If $c_1 < 0$ and $c_2 < 0$.

Case 1.2.6.1 Assume $1 + a_2^2 c_1 > 0$.

For the natural basis B' such that $P_{B'B} = \begin{pmatrix} 1 & a_2 & a_3 \\ -a_2 c_1 & 1 & 0 \\ \frac{-a_3 c_2}{1+c_1 a_2^2} & \frac{-a_3 a_2 c_2}{1+c_1 a_2^2} & 1 \end{pmatrix}$, we obtain that $M_{B'} =$
$\begin{pmatrix} 1 + a_2^2 c_1 + a_3^2 c_2 & 0 & 0 \\ c_1(1 + c_1 a_2^2) & 0 & 0 \\ \frac{c_2(1+a_2^2 c_1 + a_3^2 c_2)}{(1+c_1 a_2^2)} & 0 & 0 \end{pmatrix}$.

Case 1.2.6.1.1 Assume $1 + a_2^2 c_1 + a_3^2 c_2 > 0$.
Now, consider the natural basis $B'' = \{f_1, f_2, f_3\}$ such that

$$P_{B''B'} = \begin{pmatrix} \frac{1}{1+a_2^2 c_1 + a_3^2 c_2} & 0 & 0 \\ 0 & \frac{1}{\sqrt{-c_1(1+c_1 a_2^2)(1+a_2^2 c_1 + a_3^2 c_2)}} & 0 \\ 0 & 0 & \frac{\sqrt{1+c_1 a_2^2}}{\sqrt{-c_2}(1+a_2^2 c_1 + a_3^2 c_2)} \end{pmatrix}$$

and the structure matrix is: $M_{B''} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$, which has appeared above.

Case 1.2.6.1.2 Assume $1 + a_2^2 c_1 + a_3^2 c_2 < 0$.
Now, consider the natural basis $B'' = \{f_1, f_2, f_3\}$ such that

$$P_{B''B'} = \begin{pmatrix} \frac{1}{1+a_2^2 c_1 + a_3^2 c_2} & 0 & 0 \\ 0 & \frac{1}{\sqrt{c_1(1+c_1 a_2^2)(1+a_2^2 c_1 + a_3^2 c_2)}} & 0 \\ 0 & 0 & \frac{\sqrt{1+c_1 a_2^2}}{\sqrt{-c_2}(1+a_2^2 c_1 + a_3^2 c_2)} \end{pmatrix}$$

and the structure matrix is: $M_{B''} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$, which has already appeared.

Case 1.2.6.2 Assume $1 + a_2^2 c_1 < 0$.

For the natural basis B' such that $P_{B'B} = \begin{pmatrix} 1 & a_2 & a_3 \\ -a_2 c_1 & 1 & 0 \\ \frac{-a_3 c_2}{1+c_1 a_2^2} & \frac{-a_3 a_2 c_2}{1+c_1 a_2^2} & 1 \end{pmatrix}$, we obtain that $M_{B'} =$
$\begin{pmatrix} 1 + a_2^2 c_1 + a_3^2 c_2 & 0 & 0 \\ c_1(1 + c_1 a_2^2) & 0 & 0 \\ \frac{c_2(1+a_2^2 c_1 + a_3^2 c_2)}{(1+c_1 a_2^2)} & 0 & 0 \end{pmatrix}$.

Case 1.2.6.2.1 Assume $1 + a_2^2 c_1 + a_3^2 c_2 > 0$.

Now, consider the natural basis $B'' = \{f_1, f_2, f_3\}$ such that

$$P_{B''B'} = \begin{pmatrix} \frac{1}{1+a_2^2c_1+a_3^2c_2} & 0 & 0 \\ 0 & \frac{1}{\sqrt{c_1(1+c_1a_2^2)(1+a_2^2c_1+a_3^2c_2)}} & 0 \\ 0 & 0 & \frac{\sqrt{-(1+c_1a_2^2)}}{\sqrt{-c_2}(1+a_2^2c_1+a_3^2c_2)} \end{pmatrix}$$

and the structure matrix is: $M_{B''} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, which has already appeared.

Case 1.2.6.2.2 Assume $1 + a_2^2c_1 + a_3^2c_2 < 0$.
Now, consider the natural basis $B'' = \{f_1, f_2, f_3\}$ such that

$$P_{B''B'} = \begin{pmatrix} \frac{1}{1+a_2^2c_1+a_3^2c_2} & 0 & 0 \\ 0 & \frac{1}{\sqrt{-c_1(1+c_1a_2^2)(1+a_2^2c_1+a_3^2c_2)}} & 0 \\ 0 & 0 & \frac{\sqrt{-(1+c_1a_2^2)}}{\sqrt{-c_2}(1+a_2^2c_1+a_3^2c_2)} \end{pmatrix}$$

and the structure matrix is:

$$M_{B''} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Case 1.2.7 Suppose that $c_1 \neq 0$, $c_2 \neq 0$ and $1 + a_2^2c_1 = 0$.
Then $a_2a_3c_1c_2 \neq 0$ and so $c_1 = -\frac{1}{a_2^2}$. For B' we have

$$M_{B'} = \begin{pmatrix} a_3^2c_2 & 0 & 0 \\ \frac{c_2}{a_3^2} & 0 & 0 \\ -a_3^2c_2 & 0 & 0 \end{pmatrix}.$$

Considering natural basis $B'' = \{\frac{1}{a_3^2c_2}e_1, \frac{1}{c_2}e_2, \frac{1}{a_3^2c_2}e_3\}$ we obtain $M_{B''} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$. Which has already appeared.

Case 1.2.8 Suppose that $c_1 \neq 0$, and $c_2 = 0$.

Considering the natural basis $B'' = \{e_1, e_3, e_2\}$ we obtain $M_{B''} = \begin{pmatrix} 1 & a_3 & a_2 \\ 0 & 0 & 0 \\ c_1 & a_3c_1 & a_2c_1 \end{pmatrix}$, and we are in the same conditions as in Case 1.1.1.2.

Case 2 Suppose that $a_1 = 0$.
The structure matrix of the evolution algebra is

$$M_B = \begin{pmatrix} 0 & a_2 & a_3 \\ 0 & a_2c_1 & a_3c_1 \\ 0 & a_2c_2 & a_3c_2 \end{pmatrix}.$$

Necessarily there exists $i \in \{2, 3\}$ such that $a_i \neq 0$. Without loss in generality we assume $a_2 \neq 0$.

Case 2.1 Assume $c_1 \neq 0$. Consider the natural basis $B'' = \{e_2, e_3, e_1\}$. Then $M_{B''} =$

$\begin{pmatrix} a_2c_1 & a_3c_1 & 0 \\ a_2c_2 & a_3c_2 & 0 \\ 1 & a_3 & 0 \end{pmatrix}$ and we are in the same conditions as in Case 1.

Case 2.2 If $c_1 = 0$.

Case 2.2.1 Assume $c_2a_3 \neq 0$.

Taking the natural basis $B'' = \{e_3, e_2, e_1\}$, we obtain $M_{B''} = \begin{pmatrix} a_3c_2 & a_2c_2 & 0 \\ 0 & 0 & 0 \\ a_3 & a_2 & 0 \end{pmatrix}$ and we are in the same conditions as in the Case1.

Case 2.2.2 Suppose that $c_2a_3 = 0$.

Case 2.2.2.1 Assume $c_2 = 0$.
Take the natural basis $B' = \{a_2e_2 + a_3e_3, \frac{1}{a_2}e_3, e_1\}$. Then

$$M_{B'} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Note that E^2 has the extension property.

Case 2.2.2.2 Assume $c_2 > 0$.
Then $a_3 = 0$. For $B' = \{a_2e_2, e_1, \frac{1}{\sqrt{c_2}}e_3\}$ we have

$$M_{B'} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Case 2.2.2.3 Assume $c_2 < 0$.
Then $a_3 = 0$. For $B' = \{a_2e_2, e_1, \frac{1}{\sqrt{-c_2}}e_3\}$ we have

$$M_{B'} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Thus we have proved the following theorem.

Theorem 3.1. *Any three dimensional real evolution algebra E with $dim(E^2) = 1$ is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$E_1 : \begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, E_2 : \begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix}, E_3 : \begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ -1 & -1 & 0 \end{pmatrix}, E_4 : \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$E_5 : \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, E_6 : \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, E_7 : \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$E_8 : \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, E_9 : \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, E_{10} : \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$E_{11} : \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, E_{12} : \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, E_{13} : \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Remark 3.1. *One can classify real three-dimensional evolution algebras in case $dim(E^2) \neq 1$. But it will contain too long cases and subcases.*

4. Approximation of three-dimensional evolution algebras ($dim(E^2) = 1$)

In this section for the evolution algebras $E_i, i = \overline{1,13}$ from Theorem 3.1 we will construct evolution algebras corresponding to fixed points of the operator F.

Let E be three dimensional evolution algebra with the matrix $(a_{ij}), i, j \in \{1, 2, 3\}$. We will rewrite the operator F for this evolution algebra as:

$$F : \begin{cases} x_1' = a_{11}x_1^2 + a_{21}x_2^2 + a_{31}x_3^2, \\ x_2' = a_{12}x_1^2 + a_{22}x_2^2 + a_{32}x_3^2, \\ x_3' = a_{13}x_1^2 + a_{23}x_2^2 + a_{33}x_3^2. \end{cases}$$

Jacobian of the operator F at the point x has a form

$$J_F(x) = \begin{pmatrix} 2a_{11}x_1 & 2a_{21}x_2 & 2a_{31}x_3 \\ 2a_{12}x_1 & 2a_{22}x_2 & 2a_{32}x_3 \\ 2a_{13}x_1 & 2a_{23}x_2 & 2a_{33}x_3 \end{pmatrix}.$$

Following [9] and [3] we define an evolution algebra $\widetilde{E}$ with matrix $J_F(x)$ as the matrix of structural constants.

There is no non-zero fixed point of the operator F for the evolution algebras E_i, $i \in \{1, 2, 3, 11, 12, 13\}$ and $(1; 0; 0)$ is the unique fixed point of the operator F for the evolution algebras $E_i, i = \overline{4, 10}$. So

$$J_F(1; 0; 0) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is easy to see that the evolution algebra with the matrix $J_F(1; 0; 0)$ is isomorphic to the evolution algebra E_4.

5. Acknowledgement

The author is indebted to Professors Sh.Ayupov and U.Rozikov for the useful advice and valuable comments.

The author thanks referees for their helpful comments.

References

[1] Cabrera Casado, Yolanda; Siles Molina, Mercedes; Velasco, M. Victoria, (2017), Classification of three-dimensional evolution algebras, Linear Algebra Appl. **524**, pp.68-108.

[2] Dzhumadil'daev, A.; Omirov, B. A.; Rozikov, U. A., (2014), On a class of evolution algebras of "chicken" population, Internat. J. Math. **25**, no. 8, 1450073, 19 pp.

[3] Imomkulov A.N., (2018), Isomorphicity of two dimensional evolution algebras and evolution algebras corresponding to their idempotents, Uzb.Math.Jour. **2** pp.63-73.

[4] Ladra M., Rozikov U.A., (2013), Evolution algebra of a bisexual population, Jour.Alg. **378** pp.153-172.

[5] Lyubich Y.I., (1992), Mathematical structures in population genetics, Springer-Verlag, Berlin,.

[6] Murodov Sh.N., (2014), Classification of two-dimensional real evolution algebras and dynamics of some two-dimensional chains of evolution algebras, Uzb.Mat.Jour. **2** pp.102-111.

[7] Murodov Sh.N., (2014), Classification dynamics of two-dimensional chains of evolution algebras, Internat. J. Math. **25**, no. 2, 1450012, 23 pp.

[8] Rozikov U.A., Tian J.P., (2011), Evolution algebras generated by Gibbs measures, Lobachevskii Jour. Math. **32** (4) pp.270-277.

[9] Rozikov U.A., Velasco M.V., A discrete-time dynamical system and an evolution algebra of mosquito population, J.Math.Biology. DOI 10.1007/500285-018-1307-x
[10] Tian J.P., (2008), Evolution Algebras and Their Applications, Lecture Notes in Math., 1921, (Springer, Berlin,.), 125 p.

Anvar Imomkulov is a PhD student of Institute of Mathematics, Tashkent, Uzbekistan. He graduated from National University of Uzbekistan (2010).